Solution to Assignment 12

Supplementary Problems

1. Verify that for any k-form ω, $d(d\omega) = 0$. You may work on the 3-dimensional space.

   Solution. Let f be a 0-form. The $df = f_x dx + f_y dy + f_z dz$ and

$$\begin{aligned} d^2 f = d(df) &= df_x \wedge dx + df_y \wedge dy + df_z \wedge dz \\ &= f_{xy} dy \wedge dx + f_{xz} dz \wedge dx + f_{yx} dx \wedge dy + f_{yz} dz \wedge dy + f_{zx} dx \wedge dz + f_{zy} dy \wedge dz \\ &= 0 \ , \end{aligned}$$

   by antisymmetry.

   Let $\omega = f dx + g dy + h dz$ be a 1-form. Then

$$df = f_y dy \wedge dx + f_z dz \wedge dx + g_x dx \wedge dy + g_z dz \wedge dy + h_x dx \wedge dz + h_y dy \wedge dz,$$

   and

$$\begin{aligned} d^2\omega = d(d\omega) &= f_{yz} dz \wedge dy \wedge dx + f_{zy} dy \wedge dz \wedge dx + g_{xz} dz \wedge dx \wedge dy \\ &\quad + g_{zx} dx \wedge dz \wedge dy + h_{xy} dy \wedge dx \wedge dz + h_{yx} dx \wedge dy \wedge dz \\ &= 0 \ , \end{aligned}$$

   by antisymmetry.

   For any k-form with $k \geq 2$ its twice exterior differentiation is a $k+2$-form which must vanish in a three dimensional space.

2. Verify (a) $\nabla \times \nabla \Phi = \mathbf{0}$, and (b) $\nabla \cdot \nabla \times \mathbf{A} = 0$ for any function Φ and vector field $\mathbf{A}$.

   Solution. Straightforward computations. The first formula says a gradient vector field is curl free and the second formula says a curl vector field is divergence free. A fundamental result on vector fields is the Helmholtz decomposition theorem: Any vector field $\mathbf{F}$ can be written as

$$\mathbf{F} = \nabla \Phi + \nabla \times \mathbf{A} \ ,$$

   for some function Φ and vector field $\mathbf{A}$. In other words, it can be expressed as the sum of a curl-free and a divergence-free vector fields.

   The converse question for (a) is: When a vector field satisfies $\nabla \times \mathbf{F} = 0$, does it exist some function Φ such that $\mathbf{F} = \nabla \Phi$? We know that it is true when the underlying space of the vector field is simply-connected.

   The converse question for (b) is: When a vector field satisfies $\nabla \cdot \mathbf{F} = 0$, does it exist some vector field $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$? It is true when the vector field is defined in a star-shaped region.